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$\mathscr{P}_1$-Hereditary Artin Algebras

Lidia Angeleri Hügel*

Mathematisches Institut der Universität, Theresienstrasse 39, D-80333 Munich, Germany

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1. INTRODUCTION

Let Λ be an artin algebra, mod Λ the category of all finitely generated right Λ-modules, and ind Λ the full subcategory of mod Λ consisting of the indecomposables. Auslander and Smalø proved in [7] that there exist uniquely determined full subcategories $\mathscr{P}_0, \mathscr{P}_1, \ldots, \mathscr{P}_\infty$ of ind Λ having the following properties:

(1) $\mathscr{P}_n$ is closed under isomorphic images for each n.

(2) $\bigcup_{i=0}^{\infty} \mathscr{P}_i = \text{ind}\,\Lambda$ and $\mathscr{P}_i \cap \mathscr{P}_j = \varnothing$ when $i \neq j$.

(3) $\mathscr{P}_n$ has only a finite number of non-isomorphic objects for each $n < \infty$.

(4) Given $n < \infty$, every module in $\bigcup_{i=n}^{\infty} \mathscr{P}_i$ is an epimorphic image of a direct sum of modules in $\mathscr{P}_n$.

(5) Given $n < \infty$ and $C \in \mathscr{P}_n$, an epimorphism $X \to C$ is splittable whenever $X \in \text{mod}\,\Lambda$ does not have any direct summand in $\bigcup_{i=0}^{n-1} \mathscr{P}_i$.

The collection $\{\mathscr{P}_0, \mathscr{P}_1, \ldots, \mathscr{P}_\infty\}$ is called the preprojective partition of mod Λ, and the modules in $\bigcup_{n \in \mathbb{N}_0} \mathscr{P}_n$ are said to be preprojective.

Obviously, $\mathscr{P}_0$ consists of the indecomposable projective modules. The objects in $\mathscr{P}_1$ can be described as follows. An indecomposable non-projective module C is in $\mathscr{P}_1$ if and only if there is an irreducible morphism $P \to C$ with P projective [7, 10.5]. For $n > 1$, however, we only know that if C is in $\mathscr{P}_n$, there is a chain of irreducible morphisms starting at a projective module and ending at C, while the converse is not true in general [7, 8.3]. In other words, $\bigcup_{n \in \mathbb{N}_0} \mathscr{P}_n$ is not always closed under irreducible successors.

* E-mail address: angeleri@rz.mathematik.uni-muenchen.de.

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Auslander and Smalø showed that $\bigcup_{n \in \mathbb{N}_0} \mathscr{P}_n$ is closed under irreducible successors and predecessors if and only if the functor $\mathrm{Hom}_\Lambda(\ ,\Lambda)$ is of finite length, thus in particular if Λ is hereditary [7, 9.15 and 9.16]. If Λ is stably equivalent to a hereditary algebra, then we know from Platzeck's work that $\bigcup_{n \in \mathbb{N}_0} \mathscr{P}_n$ is closed under irreducible successors, though $\mathrm{Hom}_\Lambda(\ ,\Lambda)$ need not have finite length. More precisely, we have the following description of the preprojective modules.

THEOREM 1.1 [15, 2.7; 7, 8.3 and 10.2]. *Let Λ be an artin algebra stably equivalent to a hereditary algebra. The following statements are equivalent for an indecomposable non-projective module C.*

(a) *C is preprojective.*

(b) *There is a chain of irreducible morphisms of indecomposable modules $P = C_0 \to C_1 \to \cdots \to C_r = C$ with P projective.*

(c) *There is an integer $n > 0$ such that $(\mathrm{DTr})^n C$ is torsionless.*

(d) *There are only a finite number of non-isomorphic indecomposable modules X such that $\underline{\mathrm{Hom}}_\Lambda(X, C) \neq 0$.*

In this paper we study another class of algebras containing the hereditary as well as algebras where $\mathrm{Hom}_\Lambda(\ ,\Lambda)$ is not of finite length, and we show that Platzeck's description of the preprojective modules applies also in our case. These algebras, which we call *$\mathscr{P}_1$-hereditary*, are defined by the following property: If C is a module in $\mathscr{P}_1$, then every indecomposable module X with a non-zero morphism $X \to C$ is in $\mathscr{P}_0 \cup \mathscr{P}_1$.

We prove that the characterization of the preprojectives in Theorem 1.1 holds for $\mathscr{P}_1$-hereditary algebras by generalizing an algorithm to compute the preprojective partition of a hereditary artin algebra given by Todorov in [18]. Hereby we need weaker assumptions than those used by Coelho for his generalization of Todorov's result [10]. There is also another algorithm to determine the modules in $\mathscr{P}_n$ over a hereditary artin algebra, which was given by Zacharia in [19]. It can be extended to $\mathscr{P}_1$-hereditary artin algebras as well.

The mentioned results basically depend on the fact that every indecomposable torsionless module over a $\mathscr{P}_1$-hereditary artin algebra either is in $\mathscr{P}_0 \cup \mathscr{P}_1$ or is the socle of some indecomposable projective injective module with each of its non-simple submodules being projective. We will see that this property is essential also in another context. It will enable us to relate the $\mathscr{P}_1$-hereditary artin algebras to the algebras with $\mathrm{add}(\mathscr{P}_0 \cup \mathscr{P}_1)$ closed under submodules. The tool will be a stable equivalence modulo $\mathscr{P}_0 \cup \mathscr{P}_1$.

In a subsequent paper we will further show that every $\mathscr{P}_1$-hereditary artin algebra has a subring with $\operatorname{add}(\mathscr{P}_0 \cup \mathscr{P}_1)$ closed under submodules and a factor ring which is a hereditary Nakayama algebra.

Throughout this paper let Λ be an artin algebra with the Jacobson radical J. By a module we always mean a finitely generated right Λ-module. For $M \in \operatorname{mod} \Lambda$ we denote by $\operatorname{ind} M$ the full subcategory of $\operatorname{ind} \Lambda$ consisting of the modules which are isomorphic to an indecomposable summand of M. Moreover, if $\mathscr{C}$ is a subcategory of $\operatorname{ind} \Lambda$ closed under isomorphic images, we write $\operatorname{add} \mathscr{C}$ for the full subcategory of $\operatorname{mod} \Lambda$ consisting of all modules isomorphic to direct summands of finite direct sums of modules in $\mathscr{C}$.

For definitions and basic results about almost split sequences, irreducible morphisms, and quivers we refer to [5; 6; 16].

2. SOME PROPERTIES

We start out with a useful characterization of $\mathscr{P}_1$-hereditary artin algebras. It still holds if we consider a right artinian ring instead of an artin algebra. Also the other results in this paper can be proven in a more general version, though we must require the existence of certain almost split sequences. For the sake of simplicity, however, we always concern ourselves with artin algebras.

THEOREM 2.1. *The following statements are equivalent.*

(a) *Λ is $\mathscr{P}_1$-hereditary.*

(b) *For each $C \in \mathscr{P}_1$ the submodule C' generated by all proper submodules of C with no non-zero projective summand is in* $\operatorname{add} \mathscr{P}_1$.

(c) *Λ satisfies both of the conditions*

(H1) *If C is a module in $\mathscr{P}_1$, then every indecomposable module X with an irreducible morphism $X \to C$ is in $\mathscr{P}_0 \cup \mathscr{P}_1$.*

(H2) *If C is a module in $\mathscr{P}_1$ and P is an indecomposable projective module with an irreducible morphism $\gamma \colon P \to C$, then every indecomposable submodule X of P with γ not vanishing on X is in $\mathscr{P}_0 \cup \mathscr{P}_1$.*

Proof. Obviously, (a) implies (c).

(c) $\Rightarrow$ (b): Consider a module $C \in \mathscr{P}_1$ with the almost split sequence $0 \to A \xrightarrow{\binom{i'}{g'}} P \oplus B \xrightarrow{(g,i)} C \to 0$, where $P = \bigoplus_{i=1}^{n} P_i$ with $P_1, \ldots, P_n \in \mathscr{P}_0$, $g = (\gamma_1, \ldots, \gamma_n)$, and $B \in \operatorname{mod} \Lambda_{\mathscr{P}_0}$. In [1, 1.4] we showed that we can compute C' as follows. We take a direct complement Q of $i'(A)$ in $\operatorname{Rad} P$ and denote by Q' the submodule of Q generated by all submodules with no non-zero projective summand. Then $Q' \in \operatorname{mod} \Lambda_{\mathscr{P}_0}$, $g|_{Q'}$ is injective, and $C' = B \oplus g(Q')$. Now, $B \in \operatorname{add} \mathscr{P}_1$ by condition (H1). Moreover, every indecomposable summand X of Q' satisfies $g(X) \neq 0$ and therefore admits a non-zero morphism $X \to C$ factoring through some γ_i. So, from condition (H2) it follows that $Q' \in \operatorname{add} \mathscr{P}_1$. Hence $C' \in \operatorname{add} \mathscr{P}_1$.

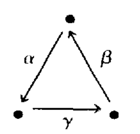
(b) $\Rightarrow$ (a): Let $C \in \mathscr{P}_1$ and $X \in \operatorname{ind} \Lambda$ with a non-zero morphism $f: X \to C$. If X is neither projective nor isomorphic to C, we have $\operatorname{Im} f \subset C'$. Observe that C' is a proper submodule of C. We can then proceed by induction on the length of C and obtain $X \in \mathscr{P}_1$. ∎

The above theorem can be employed to compute some examples.

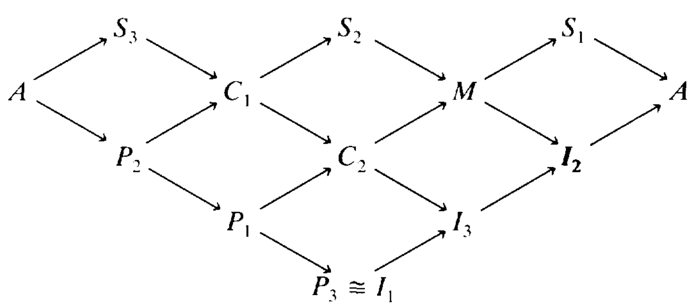
EXAMPLE 2.2. (a) First of all, we would like to compare $\mathscr{P}_1$-hereditary artin algebras with other generalizations of hereditary algebras.

We begin by observing that there are $\mathscr{P}_1$-hereditary artin algebras of infinite global dimension, for instance $k[x]/(x^2)$.

Further, we consider the Nakayama algebra Λ^* given by the quiver



with the relation $\alpha\beta\gamma = 0$. The Auslander-Reiten quiver is



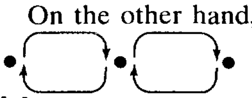
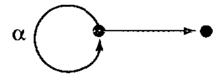
where one has to identify the two copies of A, and the preprojective partition is $\mathscr{P}_0 = \{P_1, P_2, P_3\}$, $\mathscr{P}_1 = \{C_1, C_2, I_3\}$, $\mathscr{P}_2 = \{S_2, M, I_2\}$, $\mathscr{P}_3 = \{S_1, A\}$, $\mathscr{P}_4 = \{S_3\}$, $\mathscr{P}_n = \varnothing$ for all $n > 4$. We see that Λ^* is quasi-hereditary of global dimension 2 (see [11]) and is not $\mathscr{P}_1$-hereditary.

In 4.2 we will give an example of a $\mathscr{P}_1$-hereditary artin algebra which is not stably equivalent to an I-hereditary artin algebra (see [8; 12]).

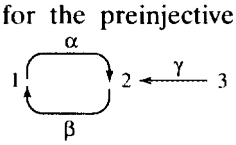
The algebra given by the quiver

$$\begin{tikzcd}
& & 3 \arrow[d] & \\
& & 4 \arrow[dr] & \\
1 & 2 \arrow[l] \arrow[ur] \arrow[dr] & & 5 \\
& & 6 \arrow[ur] & \\
& & 7 \arrow[u] &
\end{tikzcd}$$

with commutativity relation is $\mathscr{P}_1$-hereditary, l-hereditary, and not stably equivalent modulo $\mathscr{P}_0 \cup \mathscr{P}_1$ to a hereditary algebra. The last property can be established by applying [13, Theorem 1]. In fact, the category $\mathscr{I}$ of the indecomposable $\mathscr{P}_0 \cup \mathscr{P}_1$-injective modules is not open to the right in $\Gamma \setminus (\mathscr{P}_0 \cup \mathscr{P}_1)$, since there is an irreducible morphism $I_5 \to I_5/S_5$ where $I_5 \in \mathscr{I}$ and I_5/S_5 has no summand in $\mathscr{P}_0 \cup \mathscr{P}_1 \cup \mathscr{I}$.

On the other hand, the radical square zero algebra given by the quiver  is stably equivalent to a hereditary artin algebra by [4] and is not $\mathscr{P}_1$-hereditary. Finally, an l-hereditary artin algebra which is not $\mathscr{P}_1$-hereditary is given by the quiver 

with the relation $\alpha^2 = 0$.

(b) The following example shows that the definition of "$\mathscr{P}_1$-hereditary" is one-sided and does not imply the dual property for the preinjective partition. Let Λ be the algebra given by the quiver 

with the relations $\alpha\beta = 0$ and $\beta\gamma = 0$. Its Auslander–Reiten quiver is

$$\begin{tikzcd}
& & & P_3 \arrow[dr] & & S_1 \\
S_1 \arrow[dr] & & S_2 \arrow[ur] \arrow[dr] & & I_2 \arrow[ur] \arrow[dr] & \\
& P_2 \arrow[ur] \arrow[dr] & & C \arrow[ur] & & I_3 \\
& & P_1 \cong I_1 \arrow[ur] & & &
\end{tikzcd}$$

where one has to identify the two copies of S_1. The preprojective partition of mod Λ is $\mathscr{P}_0 = \{P_1, P_2, P_3\}$, $\mathscr{P}_1 = \{S_2, C, I_2\}$, $\mathscr{P}_2 = \{S_1, I_3\}$, $\mathscr{P}_n = \varnothing$ for all $n > 2$, while the preinjective partition is $\mathscr{I}_0 = \{I_1, I_2, I_3\}$, $\mathscr{I}_1 = \{P_2, P_3, C\}$, $\mathscr{I}_2 = \{S_1, S_2\}$, $\mathscr{I}_n = \varnothing$ for all $n > 2$. So, Λ is $\mathscr{P}_1$-hereditary but not $\mathscr{I}_1$-hereditary, because there is an irreducible morphism $P_2 \to S_2$ where $P_2 \in \mathscr{I}_1$ and $S_2 \in \mathscr{I}_2$. Applying the duality D, we then obtain that Λ is not left-$\mathscr{P}_1$-hereditary.

Moreover, Λ is an example of a $\mathscr{P}_1$-hereditary artin algebra where $\operatorname{add}(\mathscr{P}_0 \cup \mathscr{P}_1)$ is not closed under submodules, since there is a simple torsionless module $S_1 \in \mathscr{P}_2$. Observe finally that Λ has no preprojective component.

(c) It is well known that for an algebra of finite-representation type the number of non-empty preprojective classes may be different from the number of non-empty preinjective classes. An example of this kind was given in [2, Example 1]. The algebra there considered is $\mathscr{P}_1$-hereditary.

We have just seen that for a $\mathscr{P}_1$-hereditary artin algebra the category $\operatorname{add}(\mathscr{P}_0 \cup \mathscr{P}_1)$ need not be closed under submodules. The indecomposable projectives with a submodule not lying in $\operatorname{add}(\mathscr{P}_0 \cup \mathscr{P}_1)$, however, have a rather simple structure.

PROPOSITION 2.3. *Let Λ be $\mathscr{P}_1$-hereditary and let P be an indecomposable projective module with an indecomposable submodule S which is not in $\mathscr{P}_0 \cup \mathscr{P}_1$. Then P is uniserial and $S = \operatorname{Soc} P$ is its only non-projective submodule. Moreover, the exact sequences $0 \to PJ^{k+1} \to PJ^k \oplus PJ^{k+1}/S \to PJ^k/S \to 0$, $0 \leq k < l(P) - 1$, given by the inclusions and the natural surjections, are almost split.*

Proof. Obviously, P is not simple and there is an irreducible morphism $g: P \to C$ with $C \in \mathscr{P}_1$. Since Λ is $\mathscr{P}_1$-hereditary we have $g(S) = 0$. By [6, 4.6] it follows that C is isomorphic to P/S with the almost split sequence $0 \to A \xrightarrow{\binom{-\iota'}{\pi'}} P \oplus A/S \xrightarrow{(\pi, \iota)} P/S \to 0$, where A is a summand of Rad P such that Soc $A = S$ is simple and ι, ι' and π, π' are the inclusions and the natural surjections, respectively.

We claim $A = \operatorname{Rad} P$. Assume that there is an indecomposable summand $\tilde{A}$ in the direct complement of A in Rad P. Then we get an almost split sequence $0 \to \tilde{A} \xrightarrow{\binom{i'}{\tilde{g}'}} P \oplus B \xrightarrow{(\tilde{g}, i)} \tilde{C} \to 0$, where i' is the inclusion and $\tilde{C} \in \mathscr{P}_1$. Again, we have $\tilde{g}(S) = 0$; hence $S = \operatorname{Ker} \tilde{g}$. By [1, 1.5] we know $\operatorname{Ker} \tilde{g} = \operatorname{Soc} i'(\tilde{A})$. Thus $S = \operatorname{Soc} A \cap \operatorname{Soc} \tilde{A}$, contradicting $A \cap \tilde{A} = 0$.

Our claim implies that Soc $P = S$ is simple. Further, any non-projective submodule U of P must be equal to S. Indeed, since P/S is in $\mathscr{P}_1$ and Λ

is $\mathscr{P}_1$-hereditary, we have $U/S \in \operatorname{add} \mathscr{P}_1$. Thus the natural surjection $U \to U/S$ is a splittable epimorphism and $S = U$.

Now it follows immediately that P is uniserial. The last statement can be proven by induction applying [6, 4.6]. ∎

As an immediate consequence of [6, 4.5 and 4.10] we have the following.

LEMMA 2.4. *Every indecomposable projective module P such that the natural surjection $P \to P/\operatorname{Soc} P$ is irreducible either is injective or lies in* ind J.

We now describe the $\mathscr{P}_1$-hereditary artin algebras by means of their torsionless modules.

THEOREM 2.5. *Λ is $\mathscr{P}_1$-hereditary if and only if the following conditions are satisfied.*

(J) *If A is a module in* ind J, *then also every indecomposable non-injective module X with an irreducible morphism $X \to A$ is in* ind J.

(T) *Every indecomposable torsionless module which is not in $\mathscr{P}_0 \cup \mathscr{P}_1$ is simple and is the only non-projective submodule of some indecomposable projective injective module.*

Proof. (i) By [7, 10.5] we know that an indecomposable module X is in ind J if and only if X is non-injective and $\operatorname{TrD} X \in \mathscr{P}_1$. With this in mind, we can easily verify that condition (J) is equivalent to condition (H1) in Theorem 2.1.

(ii) We show that every $\mathscr{P}_1$-hereditary artin algebra satisfies condition (T). Let $S \in \operatorname{ind} \Lambda \setminus (\mathscr{P}_0 \cup \mathscr{P}_1)$ be torsionless. If S is contained in $P = \bigoplus_{i=1}^{n} P_i$ where $P_1, \ldots, P_n \in \mathscr{P}_0$, then for every $1 \leq i \leq n$ such that the projection $\operatorname{pr}_i : \bigoplus_{i=1}^{n} P_i \to P_i$ does not vanish on S it holds by Proposition 2.3 that $\operatorname{Soc} P_i$ is simple and the only non-projective submodule of P_i; hence $\operatorname{pr}_i(S) = \operatorname{Soc} P_i$. We infer that S is simple and that there is some $P \in \mathscr{P}_0$ with a monomorphism $S \to P$. If we choose P of maximal length with respect to this property, we obtain that P is injective by Lemma 2.4 and Proposition 2.3.

(iii) We apply Theorem 2.1. The "only if" part of our statement is then established by (i) and (ii). Moreover, to prove the "if" part, we have only to verify condition (H2). So, let us consider a module $C \in \mathscr{P}_1$ and an irreducible morphism $\gamma : P \to C$ with $P \in \mathscr{P}_0$. Assume that P has a submodule $S \in \operatorname{ind} \Lambda \setminus (\mathscr{P}_0 \cup \mathscr{P}_1)$ such that $\gamma(S) \neq 0$. By (T) we know that S is simple and is the only non-projective submodule of an indecomposable projective injective module I. We then have a commutative diagram

$$\begin{tikzcd} S \subset P \arrow[r, "\gamma"] \arrow[d] & C \arrow[dl, "h"] \\ I & \end{tikzcd}$$

where $\operatorname{Im} h \in \operatorname{mod} \Lambda_{\mathscr{P}_0}$; hence $\operatorname{Im} h = S$. But this means that the inclusion $S \subset P$ is splittable, a contradiction. ∎

Recall that a simple, non-projective, non-injective module S over an artin algebra is called a *node* if the almost split sequence beginning at S has a projective middle-term (see [12]). From the above results we get the following.

COROLLARY 2.6. *Let Λ be $\mathscr{P}_1$-hereditary.*

(a) *Every indecomposable torsionless module not lying in $\mathscr{P}_0 \cup \mathscr{P}_1$ is a node.*

(b) *Every indecomposable module contained in the Jacobson radical of some projective module is in* ind J.

Proof. (a) follows immediately from Theorem 2.5 and Proposition 2.3.

(b) Let P be projective and A an indecomposable submodule of Rad P. By (a) it suffices to consider the case $A \in \mathscr{P}_0 \cup \mathscr{P}_1$. We claim that there is a chain of irreducible morphisms $A = A_r \to \cdots \to A_1 \to A_0$ where $A_0 \in \mathscr{P}_0$ and $A_1, \ldots, A_{r-1} \in \mathscr{P}_0 \cup \mathscr{P}_1$. First of all, we have a non-zero non-isomorphism $f: A \to A_0$ for some indecomposable summand A_0 of P. Next, we recall that every $X \in \mathscr{P}_0 \cup \mathscr{P}_1$ admits a minimal right almost split morphism $Y \to X$ where Y either lies in $\operatorname{add}(\mathscr{P}_0 \cup \mathscr{P}_1)$ or is a node. Moreover, $\operatorname{Hom}_\Lambda(A, S) = 0$ for every node $S \notin \mathscr{P}_1$. In fact, every $h \in \operatorname{Hom}_\Lambda(A, S)$ factors through an injective module by [5, 5.5], and therefore we have $h = h'|_A$ for some $h' \in \operatorname{Hom}_\Lambda(P, S)$, which implies $\operatorname{Im} h \subset h'(PJ) = 0$ (these arguments are borrowed from the proof of [15, 5.4]). So, by employing the lemma of Harada and Sai, we can easily conclude that f can be written as a sum of compositions of irreducible morphisms between modules in $\mathscr{P}_0 \cup \mathscr{P}_1$. This completes the proof of our claim.

Condition (J) in Theorem 2.5 now yields for each $1 \leq i \leq r$ that A_i either lies in ind J or is injective (observe that A_i must be projective if it is injective). In particular, A is then in ind J. ∎

EXAMPLE 2.7. (a) Over a $\mathscr{P}_1$-hereditary artin algebra, nodes can occur in each $\mathscr{P}_n$ with $1 \leq n \leq \infty$. We considered an algebra with a node in $\mathscr{P}_2$

in Example 2.2(b). Further, for every $n \in \mathbb{N}$ the $\mathscr{P}_1$-hereditary algebra Λ_n given by the quiver $1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2 \xrightarrow{\gamma} 3 \cdots n+1 \longrightarrow n+2 \longleftarrow n+3$ with the relations $\alpha\beta = \beta\alpha = 0$ and $\gamma\alpha = 0$ has the nodes $S_1 \in \mathscr{P}_1$ and $S_2 \in \mathscr{P}_{n+2}$. Finally, we will give an example of a $\mathscr{P}_1$-hereditary artin algebra with a node which is not preprojective in 4.2.

(b) We saw in the proof of Theorem 2.5 that conditions (H1) and (J) are equivalent and that condition (T) implies (H2). However, (T) and (H2) are not equivalent, since the radical cube zero algebra given by the quiver $\bullet \rightleftarrows \bullet$ satisfies (H2) but not (T). Moreover, the radical square zero algebra given by the quiver $\bullet \rightleftarrows \bullet \rightleftarrows \bullet$ satisfies (H1) but not (H2), hence (J) but not (T), and the algebra given by the quiver $\bullet \xrightarrow[\alpha]{} \bullet \xrightarrow[\beta]{} \bullet \xrightarrow[\gamma]{} \bullet$ with the relation $\gamma\beta\alpha = 0$ satisfies (T) but not (J), hence (H2) but not (H1).

3. A STABLE EQUIVALENCE

Auslander and Reiten proved in [4] that an artin algebra Λ is stably equivalent to a hereditary artin algebra if and only if it satisfies the following conditions.

(1) Every indecomposable submodule of an indecomposable projective module is projective or simple.

(2) If S is a simple non-projective submodule of a projective module, then S is a factor of an injective module.

Later, Platzeck [14] gave a direct construction for a hereditary artin algebra stably equivalent to Λ provided (1) and (2) are satisfied.

The same techniques were then employed by Martinez [12] to show that every artin algebra with nodes is stably equivalent to an artin algebra without nodes.

By our Corollary 2.6 and by the fact that nodes are always factors of injective modules [5, 5.6], we now see that every $\mathscr{P}_1$-hereditary artin algebra satisfies two conditions which are quite similar to the Auslander–Reiten conditions, namely,

(1) Every indecomposable submodule of an indecomposable projective module is in $\mathscr{P}_0 \cup \mathscr{P}_1$ or simple.

(2) If S is a simple submodule of a projective module and is not in $\mathscr{P}_0 \cup \mathscr{P}_1$, then S is a factor of an injective module.

This suggests that $\mathscr{P}_1$-hereditary artin algebras could admit stable equivalence in a way analogous to that considered by Platzeck and Martinez. In fact, we are going to show that every $\mathscr{P}_1$-hereditary artin algebra is stably equivalent modulo $\mathscr{P}_0 \cup \mathscr{P}_1$ to a $\mathscr{P}_1$-hereditary artin algebra with all indecomposable torsionless modules lying in $\mathscr{P}_0 \cup \mathscr{P}_1$.

Throughout this section let Λ be a $\mathscr{P}_1$-hereditary artin algebra. For every module $M \in \operatorname{mod} \Lambda$ we denote by $T(M)$ the sum of all simple, torsionless submodules of M which are not in $\mathscr{P}_0 \cup \mathscr{P}_1$. Further, we set $\mathfrak{a} = T(\Lambda)$ and call $\mathfrak{b}$ the right annihilator of $\mathfrak{a}$ in Λ. Since $\mathfrak{a} \subset J \subset \mathfrak{b}$, the two-sided ideal $\mathfrak{a}$ is a $\Lambda/\mathfrak{a}$-$\Lambda/\mathfrak{b}$-bimodule. We put $\Gamma = \begin{pmatrix} \Lambda/\mathfrak{a} & \mathfrak{a} \\ 0 & \Lambda/\mathfrak{b} \end{pmatrix}$ and look at $\operatorname{mod} \Gamma$ as the category of all triples (A, B, f) where $A \in \operatorname{mod} \Lambda/\mathfrak{a}$, $B \in \operatorname{mod} \Lambda/\mathfrak{b}$, and $f \in \operatorname{Hom}_{\Lambda/\mathfrak{b}}(A \otimes_{\Lambda/\mathfrak{a}} \mathfrak{a}, B)$. Then we have a functor $F : \operatorname{mod} \Lambda \to \operatorname{mod} \Gamma$ defined on $M \in \operatorname{mod} \Lambda$ by $F(M) = (M/M\mathfrak{a}, M\mathfrak{a}, f_M)$ where $f_M : M/M\mathfrak{a} \otimes \mathfrak{a} \to M\mathfrak{a}$ is the multiplication map, and on $g \in \operatorname{Hom}_\Lambda(M, N)$ by $F(g) = (\bar{g}, g')$ where $\bar{g} : M/M\mathfrak{a} \to N/N\mathfrak{a}$ and $g' : M\mathfrak{a} \to N\mathfrak{a}$ are the morphisms induced by g.

We will now show that F provides a stable equivalence modulo $\mathscr{P}_0 \cup \mathscr{P}_1$ and that $\operatorname{add}(\mathscr{P}_0(\Gamma) \cup \mathscr{P}_1(\Gamma))$ is closed under submodules. But first we need some preliminary results.

LEMMA 3.1. (a) *For every $M \in \operatorname{mod} \Lambda$ it holds that $M\mathfrak{a} \subset T(M)$. If M is projective, then $M\mathfrak{a} = T(M)$.*

(b) *Let $M \in \operatorname{mod} \Lambda$ be indecomposable. $M\mathfrak{a} \neq 0$ if and only if M is projective and contains an indecomposable module S which is not in $\mathscr{P}_0 \cup \mathscr{P}_1$. Then $S = M\mathfrak{a} = \operatorname{Soc} M$.*

Proof. (a) is shown as in [15, 5.1].

(b) The "if" part of the statement follows immediately from (a). Now let $M\mathfrak{a} \neq 0$. Then we know by (a) that M has a proper submodule S which is simple torsionless and is not in $\mathscr{P}_0 \cup \mathscr{P}_1$. Further, S is the only non-projective submodule of an indecomposable projective injective module I by Theorem 2.5. So, we conclude from the commutative diagram

$$\begin{tikzcd} S \arrow[r, phantom, "\subset"] \arrow[d] & M \arrow[dl] \\ I & \end{tikzcd}$$

that M is isomorphic to a projective submodule of I. Hence $S = \operatorname{Soc} M = M\mathfrak{a}$ and the proof is complete. ∎

LEMMA 3.2. *Let $M \in \operatorname{mod} \Lambda$ be a module satisfying $M\mathfrak{a} = 0$ and let $0 \to K \xrightarrow{i} P \xrightarrow{h} M \to 0$ be a projective cover.*

(a) $K = V \oplus Q$ *where* $V \subset P\mathfrak{a}$ *and* $Q \in \operatorname{add}(\mathscr{P}_0 \cup \mathscr{P}_1)$.

(b) *Let* $\rho : K \to K/K\mathfrak{a}$ *be the canonical surjection and* $\epsilon : M \otimes \mathfrak{a} \to K/K\mathfrak{a}$, $h(p) \otimes a \mapsto \rho(pa)$. *Then* ϵ *is a splittable monomorphism with* $\operatorname{Im} \epsilon = \rho(V)$ *and the sequence* $0 \to M \otimes \mathfrak{a} \xrightarrow{\epsilon} K/K\mathfrak{a} \xrightarrow{\bar{i}} P/P\mathfrak{a} \xrightarrow{\bar{h}} M \to 0$ *is exact.*

(c) *Every indecomposable torsionless summand of* Q/QJ *lies in* $\mathscr{P}_0 \cup \mathscr{P}_1$.

Proof. (a) According to [15, 5.2], we take a decomposition $K = V \oplus Q$ where $V \in \operatorname{mod} \Lambda_{\mathscr{P}_0 \cup \mathscr{P}_1}$ and $Q \in \operatorname{add}(\mathscr{P}_0 \cup \mathscr{P}_1)$, and obtain $V \subset P\mathfrak{a}$ by Lemma 3.1.

(b) We show that ϵ is a monomorphism as in [14, 1.6]. Further, it is not hard to see that $Q \cap P\mathfrak{a} = Q\mathfrak{a} = K\mathfrak{a}$; hence $P\mathfrak{a} = V \oplus K\mathfrak{a}$. An easy computation then gives $\operatorname{Im} \epsilon = \rho(V) = \operatorname{Ker} \bar{i}$ and $K/K\mathfrak{a} = \rho(V) \oplus \rho(Q)$.

(c) Since $Q \subset \operatorname{Rad} P$, we know from the proof of Corollary 2.6(b) that $(Q, S) = 0$ for every node $S \notin \mathscr{P}_1$. Our claim then holds by Corollary 2.6(a). ∎

Let $\mathscr{S}$ be the full subcategory of $\operatorname{mod} \Gamma$ consisting of all those triples (A, B, f) such that f is an epimorphism. As Platzeck does in [14, 1.7 and 1.10], we can infer the following from 3.1 and 3.2.

PROPOSITION 3.3. *$F : \operatorname{mod} \Lambda \to \mathscr{S}$ is a full and dense functor.* ∎

For $M, N \in \operatorname{mod} \Lambda$ let $P(M, N) = \{g \in \operatorname{Hom}_\Lambda(M, N) \mid g$ factors through a projective module$\}$ and $V(M, N) = \{g \in \operatorname{Hom}_\Lambda(M, N) \mid g$ factors through a module in $\operatorname{add}(\mathscr{P}_0 \cup \mathscr{P}_1)\}$.

LEMMA 3.4. *Let* $M, N \in \operatorname{mod} \Lambda$.

(a) *For every* $g \in \operatorname{Hom}_\Lambda(M, N)$ *it holds that* $F(g) = 0$ *if and only if* $\operatorname{Im} g \subset N\mathfrak{a}$. *If* N *is projective, then* $F(g) = 0$ *if and only if* $\operatorname{Im} g \in \operatorname{mod} \Lambda_{\mathscr{P}_0 \cup \mathscr{P}_1}$.

(b) $\{g \in \operatorname{Hom}_\Lambda(M, N) \mid F(g) = 0\} \subset P(M, N)$.

(c) If $M \in \operatorname{mod} \Lambda_{\mathscr{P}_0 \cup \mathscr{P}_1}$, *then* $V(M, N) = P(M, N)$ *and* $\operatorname{Hom}_\Lambda(M, N)/V(M, N) \cong \operatorname{Hom}_\Gamma(F(M), F(N))$.

(d) *If* $M \in \mathscr{P}_1$ *is not torsionless, then* $\operatorname{Hom}_\Lambda(M, N)/P(M, N) \cong \operatorname{Hom}_\Gamma(F(M), F(N))$.

Proof. (a) follows from Lemma 3.1.

(b) is shown as in [14, 2.1].

(c) We prove $V(M, N) \subset \{g \in \operatorname{Hom}_\Lambda(M, N) \mid F(g) = 0\}$. To this end let us consider a commutative diagram

$$\begin{tikzcd} M \arrow[rr, "g"] \arrow[dr, "h"'] & & N \\ & P \oplus C \arrow[ur, "f"'] & \end{tikzcd}$$

where $P \in \operatorname{add} \mathscr{P}_0$ and $C \in \operatorname{add} \mathscr{P}_1$. Since Λ is $\mathscr{P}_1$-hereditary and $M \in \operatorname{mod} \Lambda_{\mathscr{P}_0 \cup \mathscr{P}_1}$, we have that $\operatorname{Im} h \in \operatorname{mod} \Lambda_{\mathscr{P}_0 \cup \mathscr{P}_1}$ is a submodule of P. Then we get $F(g) = 0$ by applying (a).

From (b) it follows that $V(M, N) = P(M, N) = \{g \in \operatorname{Hom}_\Lambda(M, N) \mid F(g) = 0\}$. By Proposition 3.3 we know that F is dense, and so we obtain the second claim.

(d) By (b) we have only to show that every $g \in P(M, N)$ satisfies $F(g) = 0$. This follows from (a), because in every commutative diagram

$$\begin{tikzcd} M \arrow[rr, "g"] \arrow[dr, "h"'] & & N \\ & P \arrow[ur, "f"'] & \end{tikzcd}$$

where P is projective, the morphism h cannot be injective, and thus $\operatorname{Im} h \in \operatorname{mod} \Lambda_{\mathscr{P}_0 \cup \mathscr{P}_1}$. ∎

PROPOSITION 3.5. (a) *Let $P \in \mathscr{P}_0$ and $C \in \operatorname{ind} \Lambda \setminus \mathscr{P}_0$. A morphism $g : P \to C$ is irreducible in* $\operatorname{mod} \Lambda$ *if and only if $F(g)$ is irreducible in* $\operatorname{mod} \Gamma$.

(b) *$\mathscr{P}_0(\Gamma)$ consists of the triples of the form $F(P)$ for some $P \in \mathscr{P}_0$ and of those indecomposable triples which are not in $\mathscr{S}$. The latter are simple and have the form $(0, B, 0)$ for some $B \in$* $\operatorname{ind} \Lambda/\mathfrak{b}$.

(c) *$\mathscr{P}_1(\Gamma)$ consists of the triples of the form $F(C)$ for some $C \in \mathscr{P}_1$.*

Proof. (a) We prove the "only if" part. By Lemma 3.1 it holds that $C\mathfrak{a} = 0$; hence $F(g)$ has the form $(\bar{g}, 0) : (P/P\mathfrak{a}, P\mathfrak{a}, f_P) \to (C, 0, 0)$ and is not an isomorphism. Consider now a commutative diagram

$$\begin{tikzcd} F(P) \arrow[rr, "F(g)"] \arrow[dr, "\tau"'] & & F(C) \\ & (A, B, f) \arrow[ur, "\sigma"'] & \end{tikzcd}$$

Since we always have a decomposition $(A, B, f) = (A, \operatorname{Im} f, f) \oplus (0, B', 0)$ we can assume $(A, B, f) \in \mathscr{S}$. Then by Proposition 3.3, there are $M \in \operatorname{mod} \Lambda$, $t \in \operatorname{Hom}_\Lambda(P, M)$, and $s \in \operatorname{Hom}_\Lambda(M, C)$ such that $F(M) =$

(A, B, f), $F(t) = \tau$, and $F(s) = \sigma$. So, we have $F(st - g) = 0$, which means $st = g$ by Lemma 3.4. But g is irreducible and thus it follows that τ is a splittable monomorphism or σ is a splittable epimorphism. The proof for the "if" part is obvious.

(b) is well known (see [14, Sect. 2]).

(c) follows easily from (a), (b), and Proposition 3.3. ∎

We are now ready to prove our announced result.

THEOREM 3.6. *Λ is stably equivalent modulo $\mathscr{P}_0 \cup \mathscr{P}_1$ to Γ. Moreover, the category* add $(\mathscr{P}_0(\Gamma) \cup \mathscr{P}_1(\Gamma))$ *is closed under submodules.*

Proof. We first show that $\operatorname{add}(\mathscr{P}_0(\Gamma) \cup \mathscr{P}_1(\Gamma))$ is closed under submodules. It suffices to consider indecomposable modules, say $Y \in \mathscr{P}_0(\Gamma) \cup \mathscr{P}_1(\Gamma)$ and a submodule $X \in \operatorname{ind} \Gamma$. Further, we can assume X and Y in $\mathscr{S}$, since the indecomposable Γ-modules not lying in $\mathscr{S}$ are simple projective (3.5). Then by Proposition 3.3 and 3.5, there are $M \in \operatorname{ind} \Lambda$ and $N \in \mathscr{P}_0 \cup \mathscr{P}_1$ such that $F(M) = X$ and $F(N) = Y$. If M were not in $\mathscr{P}_0 \cup \mathscr{P}_1$, we would obtain $\operatorname{Hom}_\Gamma(X, Y) \cong \operatorname{Hom}_\Lambda(M, N)/V(M, N) = 0$ by Lemma 3.4. Hence $M \in \mathscr{P}_0 \cup \mathscr{P}_1$, and thus $X \in \mathscr{P}_0(\Gamma) \cup \mathscr{P}_1(\Gamma)$ by Proposition 3.5.

Now it follows from Lemma 3.4 that F induces an isomorphism $\operatorname{Hom}_\Lambda(M, N)/V(M, N) \to \operatorname{Hom}_\Gamma(F(M), F(N))/V_\Gamma(F(M), F(N))$ for all $M, N \in \operatorname{mod} \Lambda_{\mathscr{P}_0 \cup \mathscr{P}_1}$, where $V_\Gamma(F(M), F(N))$ is defined like $V(M, N)$. Moreover, we know by Proposition 3.3 and 3.5 that $F : \operatorname{mod} \Lambda_{\mathscr{P}_0 \cup \mathscr{P}_1} \to \operatorname{mod} \Gamma_{\mathscr{P}_0(\Gamma) \cup \mathscr{P}_1(\Gamma)}$ is a dense functor. Thus F provides a stable equivalence modulo $\mathscr{P}_0 \cup \mathscr{P}_1$. ∎

The next proposition describes the case where Γ is even hereditary.

PROPOSITION 3.7. *The following statements are equivalent.*

(a) *Γ is a hereditary artin algebra.*

(b) *Γ is a hereditary artin algebra stably equivalent to Λ.*

(c) *There are no torsionless modules in $\mathscr{P}_1$.*

Proof. (a) $\Rightarrow$ (c): Suppose that some $C \in \mathscr{P}_1$ is torsionless. By Corollary 2.6 there is a module $P \in \mathscr{P}_0$ with an irreducible morphism $g : C \to P$. From (a) and Proposition 3.5 it follows that $F(g) = 0$, which means that g factors through $\operatorname{Im} g \in \operatorname{mod} \Lambda_{\mathscr{P}_0 \cup \mathscr{P}_1}$ by Lemma 3.4. But this is a contradiction.

(c) $\Rightarrow$ (b): From (c) and Lemma 3.4 it follows for all $M \in \operatorname{mod} \Lambda_{\mathscr{P}_0}$ and $N \in \operatorname{mod} \Lambda$ that $\operatorname{Hom}_\Lambda(M, N)/P(M, N) \cong \operatorname{Hom}_\Gamma(F(M), F(N))$. By Proposition 3.3 and 3.5 we infer that $\operatorname{Hom}_\Gamma(X, Y) = 0$ whenever $X \in \operatorname{mod} \Gamma_{\mathscr{P}_0(\Gamma)}$ and Y is a projective Γ-module. Thus Γ is hereditary. In particular, F then induces a stable equivalence modulo $\mathscr{P}_0$.

(b) $\Rightarrow$ (a) is obvious. ∎

EXAMPLE 3.8. (a) The algebra Λ_n, $n \in \mathbb{N}$, of Example 2.7(b) is stably equivalent to a hereditary artin algebra, but Γ is not hereditary, since condition (c) in Proposition 3.7 is not satisfied.

(b) The Nakayama algebra Λ^* of Example 2.2(a) is not stably equivalent to a hereditary artin algebra, though there are no torsionless modules in $\mathscr{P}_1$.

4. THE PREPROJECTIVE PARTITION

This section is devoted to the preprojective partition of a $\mathscr{P}_1$-hereditary artin algebra. We begin by fixing some notation. Let $\mathscr{P}_0, \mathscr{P}_1, \ldots, \mathscr{P}_\infty$ be the preprojective partition of mod Λ. For $n \in \mathbb{N}$ we set $\mathscr{P}^n = \mathscr{P}_0 \cup \cdots \cup \mathscr{P}_{n-1}$ and denote by $\operatorname{mod} \Lambda_{\mathscr{P}^n}$ the full subcategory of mod Λ consisting of all modules with no summand in $\mathscr{P}^n$. Further, we say that Λ is $\mathscr{P}_n$-hereditary if it satisfies the following property: If C is a module in $\mathscr{P}_n$, then every indecomposable module X with a non-zero morphism $X \to C$ is in $\mathscr{P}^{n+1}$.

In [18] Todorov gave an algorithm to determine the modules in the preprojective partition of a hereditary artin algebra. We are now going to prove a generalization of her result.

Todorov's algorithm has already been generalized by Coelho in [10] under both of the following hypotheses.

(I) Add $\mathscr{P}^{n+1}$ is closed under submodules.

(II) If A is a module in $\mathscr{P}^n$, then every indecomposable module X with an irreducible morphism $A \to X$ is in $\mathscr{P}^{n+1}$.

For algebras satisfying condition (I), the functor $\operatorname{Hom}_\Lambda(\ ,\Lambda)$ is of finite length, which means that the subcategory of the preprojective modules is closed under irreducible predecessors and successors (see [7, 9.15 and 9.16]).

Here we replace condition (I) by the weaker assumption that Λ is $\mathscr{P}_n$-hereditary. So, we study a class of algebras including those considered by Coelho as well as some artin algebras where $\operatorname{Hom}_\Lambda(\ ,\Lambda)$ is not of finite length (see Example 4.2).

Actually, using a different approach, R. Betzler [9] has recently obtained an even more general version of Todorov's theorem. Indeed, he only requires

(Ia) if C is a module in $\mathscr{P}_n$, then every indecomposable module X with an irreducible morphism $X \to C$ is in $\mathscr{P}^{n+1}$;

(Ib) if X and Y are indecomposable modules not lying in $\mathscr{P}^n$, then every morphism $X \to Y$ factoring through add $\mathscr{P}^n$ has a superfluous image;

together with condition (II), and it is not hard to see that $\mathscr{P}_n$-hereditary artin algebras satisfy (Ia) as well as (Ib).

Let us now prove our result.

PROPOSITION 4.1. *Let $n \in \mathbb{N}$ and let Λ be a $\mathscr{P}_n$-hereditary artin algebra satisfying condition* (II). *Then the following are true for every $m > n$.*

(a) *An indecomposable module C not lying in $\mathscr{P}^n$ is in $\mathscr{P}_m$ if and only if it is not in $\mathscr{P}_{m-1}$ and there is an irreducible morphism $X \to C$ where X is in $\mathscr{P}_{m-1}$.*

(b) *Λ is $\mathscr{P}_m$-hereditary.*

Proof. (1) We prove that an indecomposable module C is in $\mathscr{P}^{n+2}$ if there is an irreducible morphism $X \to C$ where $X \in \mathscr{P}_n$. We assume that $C \notin \mathscr{P}^{n+1}$. By the hypothesis, we have an almost split sequence of the form $0 \to A \xrightarrow{\alpha} P \oplus B \xrightarrow{\beta} C \to 0$ where $0 \neq P \in \text{add}\, \mathscr{P}_n$ and $B \in \text{mod}\, \Lambda_{\mathscr{P}^{n+1}}$. Suppose that there is $Y \in \text{mod}\, \Lambda_{\mathscr{P}^{n+1}}$ with a non-splitting epimorphism $f: Y \to C$. Then f factors through $\beta|_B$ since $\text{Hom}_\Lambda(Y, P) = 0$. It follows that $\beta|_B$ and $\text{pr}_P\, \alpha$ are epimorphisms, hence $A \in \mathscr{P}^n$. Again using the hypothesis, we obtain $B = 0$, a contradiction. Thus we conclude $C \in \mathscr{P}_{n+1}$.

(2) In particular, we infer from (1) that condition (II) still holds if we replace n by $n + 1$.

(3) Next, we investigate an almost split sequence $0 \to A \to E \to C \to 0$ where $C \in \mathscr{P}_{n+1}$. We consider a decomposition $E = P \oplus B$ where $0 \neq P \in \text{add}\, \mathscr{P}^{n+1}$ and $B \in \text{mod}\, \Lambda_{\mathscr{P}^{n+1}}$. By the hypothesis we know that $P \in \text{add}\, \mathscr{P}_n$ and therefore $A \in \mathscr{P}^{n+1}$. From (2) it then follows that $B \in \text{add}\, \mathscr{P}_{n+1}$.

(4) Let us verify the statements in the proposition for $m = n + 1$. In (1) and (3) we have just established (a). Now we prove (b). We take $C \in \mathscr{P}_{n+1}$, $X \in \text{ind}\, \Lambda$ with a non-zero morphism $f: X \to C$, and assume $X \notin \mathscr{P}_{n+1}$. This implies that f factors through the almost split sequence $0 \to A \to E \to C \to 0$. In particular, there are $C_1 \in \text{ind}\, E$ and $0 \neq f_1 \in \text{Hom}_\Lambda(X, C_1)$. By (3) we know that $C_1 \in \mathscr{P}_n \cup \mathscr{P}_{n+1}$. If $C_1 \in \mathscr{P}_{n+1}$, then f_1 factors through the almost split sequence $0 \to A_1 \to E_1 \to C_1 \to 0$. Again, we find $C_2 \in \text{ind}\, E_1$ and $0 \neq f_2 \in \text{Hom}_\Lambda(X, C_2)$, and we know $C_2 \in \mathscr{P}_n \cup \mathscr{P}_{n+1}$.

We can continue in this fashion obtaining a chain of irreducible monomorphisms $\cdots C_2 \to C_1 \to C$ in $\mathscr{P}_{n+1}$, unless some C_i occurs in $\mathscr{P}_n$. But C is artinian. So, after a finite number of steps, we must find a module $C_i \in \mathscr{P}_n$ with $\operatorname{Hom}_\Lambda(X, C_i) \neq 0$. Thus $X \in \mathscr{P}^{n+1}$ since Λ is $\mathscr{P}_n$-hereditary.

(5) Proceeding by induction, we obtain that (a) and (b) are true for all $m > n$. ∎

The following example shows that $\operatorname{Hom}_\Lambda(\ , \Lambda)$ need not have finite length when Λ is $\mathscr{P}_1$-hereditary.

EXAMPLE 4.2. Let Λ be the algebra given by the quiver

$$\begin{tikzcd}
 & & 4 & 7 \arrow[d] & \\
1 \arrow[r, "\alpha"'] & 2 \arrow[r, "\beta"'] & 3 \arrow[u] \arrow[r, "\gamma"'] & 6 \arrow[r, "\delta"'] & 5
\end{tikzcd}$$

with the relations $\beta\alpha = 0$ and $\delta\gamma = 0$.

Λ is of tame representation type, because we know by [17, Theorem 1, Nr. 29] that the algebra Λ' obtained from Λ by deleting the vertex 1 is tame, and P_1 and S_1 are the only indecomposable Λ-modules which are not Λ'-modules.

Consider the following parts of the Auslander–Reiten quiver of Λ (actually, we can conclude from the discussion below that they are two different components):

$$\begin{tikzcd}
 & & P_7 \cong I_5 \arrow[dr] & & & & \\
 & P_6 \arrow[ur] \arrow[dr] & & C_1 \arrow[dr] & & \cdots & \\
P_5 \arrow[ur] & & S_6 \arrow[ur] \arrow[dr] & & C_2 \arrow[ur] \arrow[dr] & & \cdots \\
 & & & P_3 \arrow[ur] \arrow[r] \arrow[dr] & P_2 \arrow[r] & C_4 \arrow[ur] \arrow[r] \arrow[dr] & \cdots \\
 & & P_4 \arrow[ur] & & C_3 \arrow[ur] & & \cdots
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & I_7 & & & \\
\cdots I_6 \arrow[ur] \arrow[dr] & & & & \\
\cdots I_4 \arrow[r] & I_3 \arrow[dr] & & P_1 \cong I_2 \arrow[dr] & \\
\cdots S_3 \arrow[ur] & & S_2 \arrow[ur] & & I_1
\end{tikzcd}$$

Λ is $\mathscr{P}_1$-hereditary by Theorem 2.1. Assume now that S_2 is preprojective. Then it follows from Proposition 4.1 that S_3 and S_7 are preprojective as well. Since $S_4, S_5 \in \mathscr{P}_0$ and $S_1, S_6 \in \mathscr{P}_1$, we obtain that all simple modules are preprojective, which is a contradiction to Λ being tame by [3, 4.7]. So, S_2 is a torsionless non-preprojective module, and the functor $\operatorname{Hom}_\Lambda(\ ,\Lambda)$ is not of finite length.

Moreover, S_6 is a torsionless non-projective module which is not a factor of an injective module. From [12] it then follows that Λ is not stably equivalent to an l-hereditary artin algebra.

Another algorithm to determine the modules in the preprojective partition of a hereditary artin algebra was obtained by Zacharia in [19, 4.5]. It can also be extended to $\mathscr{P}_1$-hereditary algebras, as we will now show.

Observe first that a $\mathscr{P}_1$-hereditary artin algebra is $\mathscr{P}_n$-hereditary for all $n \in \mathbb{N}$ by Proposition 4.1. In particular, every non-zero morphism $C \to D$ where $C, D \in \mathscr{P}_n$ for some $n \in \mathbb{N}$ must be injective. So, the isomorphism classes of $\mathscr{P}_n$ can be partially ordered in a way similar to [16, 2.2] by setting $C \leq D$ if $\operatorname{Hom}_\Lambda(C, D) \neq 0$. Maximal objects with respect to this order can then be determined as follows.

LEMMA 4.3. *Let Λ be $\mathscr{P}_1$-hereditary. Further, let $n \in \mathbb{N}$ and let C be a module in $\mathscr{P}_n$ which is not torsionless. Then C is maximal in $\mathscr{P}_n$ with respect to "$\leq$" if and only if there is no irreducible morphism $C \to D$ where $D \in \mathscr{P}_n$.*

Proof. We have only to prove the "if" part of our statement. We suppose that C is not maximal, choose a module $D \in \mathscr{P}_n$ of minimal length with respect to the property of admitting a non-zero non-isomorphism $f: C \to D$, and show that f is irreducible. Consider a commutative diagram

$$\begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "\alpha"'] & & D \\ & X \arrow[ur, "\beta"'] \arrow[d, "\mathrm{pr}_{X'}"'] & \\ & X' & \end{tikzcd}$$

where $X \in \operatorname{mod} \Lambda$ and X' is an indecomposable summand of X such that $\beta|_{X'} \mathrm{pr}_{X'} \alpha \neq 0$. Since Λ is $\mathscr{P}_k$-hereditary for all $1 \leq k \leq n$, it follows that $X' \in \mathscr{P}_0 \cup \mathscr{P}_n$. Further, f and $\mathrm{pr}_{X'} \alpha$ are injective. But C is not torsionless, hence $X' \in \mathscr{P}_n$, and therefore $\beta|_{X'}$ is injective as well. By length arguments we now conclude that α is a splittable monomorphism or β is a splittable epimorphism. ∎

Let us call a module $P \in \mathscr{P}_0$ maximal if it is not in ind J. Recall that $\mathscr{P}_1 = \{\mathrm{TrD}\, X | X \in \text{ind}\, J\}$ by [7, 10.5]. If the algebra is $\mathscr{P}_1$-hereditary, an analogous result for higher classes $\mathscr{P}_n$ is given by the following generalization of Zacharia's algorithm.

PROPOSITION 4.4. *Let* Λ *be* $\mathscr{P}_1$*-hereditary. For each* $n \in \mathbb{N}$ *it holds that* $\mathscr{P}_{n+1} = \{\mathrm{TrD}\, X | X$ *is non-injective, not in* ind J *and is maximal in* $\mathscr{P}_{n-1}\} \cup \{\mathrm{TrD}\, Y | Y$ *is not in* ind J *and is non-maximal in* $\mathscr{P}_n\}$.

Proof. We take a module $C \in \mathscr{P}_{n+1}$ and show that it lies in the indicated set. As we saw in Proposition 4.1, we have an almost spit sequence $0 \to A \to P \oplus B \to C \to 0$ where $0 \neq P \in \text{add}\, \mathscr{P}_n$, $B \in \text{add}\, \mathscr{P}_{n+1}$, and $A \in \mathscr{P}_{n-1} \cup \mathscr{P}_n$. Since $C \notin \mathscr{P}_1$, we have $A \notin \text{ind}\, J$. Of course, A is non-maximal if it is in $\mathscr{P}_n$. Suppose now that $A \in \mathscr{P}_{n-1}$. For $n = 1$ it then follows that A is maximal in $\mathscr{P}_0$. For $n > 1$ we obtain that A is not torsionless, because otherwise A would be in ind J by Corollary 2.6. So, we can apply Lemma 4.3 and conclude that A is maximal in $\mathscr{P}_{n-1}$.

The other inclusion is shown by similar arguments. ∎

Given a $\mathscr{P}_1$-hereditary artin algebra, Propositions 4.1 and 4.4 now enable us to prove the same characterization of the preprojective modules as was obtained by Platzeck in [15] for algebras stably equivalent to a hereditary artin algebra.

THEOREM 4.5. *Let* Λ *be* $\mathscr{P}_1$*-hereditary. The following statements are equivalent for an indecomposable non-projective module* C.

(a) *C is preprojective.*

(b) *There is a chain of irreducible morphisms of indecomposable modules* $P = C_0 \to C_1 \to \cdots \to C_r = C$ *with* P *projective.*

(c) *There is an integer* $n > 0$ *such that* $(\mathrm{DTr})^n C$ *is torsionless.*

(d) *There are only a finite number of non-isomorphic indecomposable modules* X *such that* $\mathrm{Hom}_\Lambda(X, C) \neq 0$.

(e) *There are only a finite number of non-isomorphic indecomposable modules* X *such that* $\underline{\mathrm{Hom}}_\Lambda(X, C) \neq 0$.

Proof. The proof for the equivalence of (a), (b), and (c) is straightforward. Since Λ is $\mathscr{P}_n$-hereditary for all $n \in \mathbb{N}$, we further have (a) $\Rightarrow$ (d) $\Rightarrow$ (e). Finally, (e) $\Rightarrow$ (a) holds by [7, 10.2]. ∎

We point out that the last two results cannot be extended to $\mathscr{P}_n$-hereditary artin algebras where $n > 1$.

EXAMPLE 4.6. Consider the Nakayama algebra Λ^* of Example 2.2(a). It is $\mathscr{P}_2$-hereditary but not $\mathscr{P}_1$-hereditary, and for $n = 1$ it does not even satisfy property (Ia) at the beginning of this section. Nevertheless, the algorithm in 4.1(a) holds for $n = 1$. If we look at the DTr-orbit in the top row of the Auslander–Reiten quiver, however, we see that the algorithm in 4.4 fails and that conditions (a) and (c) in Theorem 4.5 are not equivalent for Λ^*.

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